

On Energy Expenditure per Unit of the Amount of Information

A.Granik*

February 9, 2008

Abstract

It is shown that for an equilibrium state of time-symmetric system of non-relativistic strings the energy per unit of information transfer (storage, processing) obeys the Bekenstein conjecture. The result is based on a theorem due to A.Kholevo relating the physical entropy and the amount of information. Interestingly, the energy in question is the difference between the ensemble averaged energy and the Helmholtz free energy.

The problem about the energy requirements for the storage, transfer, and processing of information is one of the most important problems in the physics of information. In view of the recent keen interest (and the attendant very large body of research work) in possible realizations of quantum computers, the above problem has direct relevance to quantum systems. Therefore it seems appropriate to revisit this problem.

At the beginning of 1990's B.Schumacher proposed (and later elaborated) [1] a conjecture (a generalization of earlier proposals by Bekenstein [2] and Pendry [3]) about an existence of a quantum limit for the power requirements of a communication channel. The conjecture relates an amount of information H conveyed by a quantum channel in a time interval δt and the energy E required for the physical representation of the information in the quantum

*Department of Physics, University of the Pacific, Stockton, CA. 95211; E-mail: agranik@pacific.edu

system. In its simplest form (proposed for the first time by Bekenstein [2]) the conjecture is as follows

$$\frac{E}{H} \geq \frac{\hbar}{\delta t} \quad (1)$$

In what follows we will prove this conjecture (in a form given by Eq.1) for an equilibrium state of an isolated system. It will be shown that the energy E turns out to be the difference between the ensemble averaged energy and the Helmholtz free energy.

To proceed with our reasoning, we consider the partial action S_i for a time-symmetric system of non-relativistic strings [4] of length δ , mass density ρ , charge density ρ_e and tension κ :

$$S_i = \int_0^{\tau_0} d\tau \int_0^{L_0} \left\{ \frac{1}{2} \rho \left(\frac{\partial \vec{x}_i}{\partial \tau} \right)^2 - \frac{1}{2} \kappa \left(\frac{\partial \vec{x}_i}{\partial \lambda} \right)^2 - \rho_e \phi[\vec{x}_i(\tau, \lambda)] \right\} d\lambda \quad (2)$$

Here τ and λ are time and space coordinates defined on the string.

If the string is assumed to be "frozen-in", that is its space coordinate in the lab frame \vec{x}_i is independent of the time τ , that is $\vec{x}_i = \vec{x}_i(\lambda)$, then Eq.(2) yields:

$$S_i = -\tau_0 \int_0^{L_0} \left\{ \frac{1}{2} \kappa \left(\frac{\partial \vec{x}_i}{\partial \lambda} \right)^2 - \rho_e \phi[\vec{x}_i(\tau, \lambda)] \right\} d\lambda \quad (3)$$

Now we transform this equation into the integral over the energy of a non-relativistic particle of a mass m . This transformation is possible because, generally speaking, a time interval is not an absolute concept but is rather defined by an appropriate physical (periodic) process.

Therefore by considering a system of frozen-in strings, we can safely introduce the respective time interval, say t , as follows

$$t = \hbar \beta \quad (4)$$

where $\beta = 1/kT$, k is the Boltzmann constant, and T is the characteristic temperature whose lower limit can be taken, for example, as the Hawking temperature(see [5])

$$T_H = \frac{1}{\sqrt{\alpha'}}$$

where α' is the string length.

This allows us to relate in a very simple fashion the space variable λ to the new time variable t given by (4)

$$t = \frac{\lambda}{c} \quad (5)$$

Here c is the speed of light. As a result, Eq.(3) becomes

$$S_i = -\tau_0 \int_0^{\hbar\beta} c dt \left\{ \frac{1}{2} \frac{\kappa}{c^2} \left(\frac{d\vec{x}_i}{dt} \right)^2 + \rho_e \phi[\vec{x}_i(t)] \right\} = - \int_0^{\hbar\beta} dt \left[\frac{m}{2} \left(\frac{d\vec{x}_i}{dt} \right)^2 + V(t) \right] \quad (6)$$

where $m = \kappa\tau_0/c$ is the mass of a non-relativistic particle in an external potential $V(t) = \tau_0\rho_e\phi(t)$, as was indicated Feynman [6] and later by Chiu [4]. It must be noted that in our representation we differ significantly from both of them, since now the time t is the **physical** time, and not the meta "time" used by Feynman [6]. The same comment is true for the expression in figure brackets representing *physical* energy E_i of a non-relativistic particle of a mass m at some moment of time $0 \leq t \leq \hbar\beta$

$$E_i = \frac{m}{2} \left(\frac{d\vec{x}_i}{dt} \right)^2 + V(t),$$

and not what Feynman referred as "energy" to remind us that it was not a physical energy.

Now let us consider a system of particles at a thermal equilibrium at some characteristic temperature (in our case its lowest limit can be taken, for example, as the Hawking temperature) and introduce ϵ_i , the energy averaged over the time interval t given by (4) and(5)

$$\epsilon_i \equiv \frac{1}{\hbar\beta} \int_0^{\hbar\beta} E_i dt \quad (7)$$

The probability p_i that the system should be in the same states of energy ϵ_i as in (7) is

$$p_i = \frac{e^{-\beta\epsilon_i}}{\sum_i e^{-\beta\epsilon_i}} \equiv \frac{e^{-\beta\epsilon_i}}{Z} \quad (8)$$

where

$$Z = \sum_i e^{-\beta \epsilon_i}$$

is the partition function.

On the other hand, the entropy \mathcal{S} (a *physical* quantity with a thermodynamic meaning defined for the above statistical ensemble) is

$$\mathcal{S} = - \sum_i p_i \ln(p_i) \quad (9)$$

Upon substitution of (8) in (9) we get

$$\mathcal{S} = \beta \langle \epsilon \rangle + \ln Z \quad (10)$$

where the ensemble average $\langle \epsilon \rangle$ is

$$\langle \epsilon \rangle = \frac{\sum_i \epsilon_i e^{-\beta \epsilon_i}}{Z}$$

By using the Kholevo theorem [7] in its simplest form

$$\mathcal{S} \geq H$$

we obtain from (10)

$$\beta \langle \epsilon \rangle + \ln Z \geq H \quad (11)$$

where H is the amount of information. On the other hand, by definition

$$\ln Z \equiv -F\beta \quad (12)$$

where F is the Helmholtz free energy. Therefore (11) yields

$$\langle \epsilon \rangle - F \geq \frac{H}{\beta} \quad (13)$$

Now using equation (4) we express β in terms of the time interval δt

$$\beta = \frac{1}{kT} = \frac{\delta t}{\hbar}$$

Inserting this expression in (13) we obtain

$$\frac{\langle \epsilon \rangle - F}{H} \geq \frac{\hbar}{\delta t} \quad (14)$$

If we associate the time interval δt with a period of an oscillatory process of frequency ω , then inequality (14) becomes:

$$\frac{\langle \epsilon \rangle - F}{H} \geq \hbar \omega \quad (15)$$

Since F at constant temperature plays the part of the potential energy, the difference $\langle \epsilon \rangle - F$ plays the part of the average kinetic energy. This means that the minimum average "kinetic energy" expenditure necessary to transmit, store or process a unit (a bit, or rather a qubit) of information is exactly one quantum $\hbar\omega$.

Another consequence of inequality (14) is that with a decrease of the temperature, the characteristic time interval necessary to transmit (store, process), and the respective temporal rate (the Kolmogorov information) of information transmission through the quantum channel decreases. This is easily explained, since at lower temperatures a system tends to reside at lower energy states, and its higher states are inaccessible, unless there is a supply of an additional energy, which results in an increase of the temperature.

References

- [1] B.Schumacher, in "Information from quantum measurements", in Entropy, Complexity, and the Physics of Information, W. H. Zurek, editor (Addison-Wesley, Reading, Massachusetts, 1990), p.29;
B.Schumacher and M D. Westmoreland, quant-ph/0004045(2000)
- [2] J.D.Bekenstein, Phys.Rev., **D23**, 287 (1981)
- [3] J.B.Pendry J.Phys, **A16**, 2161 (1983)
- [4] S.-Y. Chiu, PRL, **71**, 2847 (1993)

- [5] E.Halyo, A.Rajaramajan, and L.Susskind, Phys.Lett. **B392**, 319 (1997)
- [6] R.Feynman and A.Hibbs, Quantum Mechanics and Path Integral, McGraw-Hill (New York),1965
- [7] A.S.Kholevo, Problemy Peredachi Informatzii (in Russian), v.9, p.3 (1973)